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Lions–Schechter's methods of complex interpolation and some quantitative estimates

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Abstract

Some quantitative estimates concerning multi-dimensional rotundity, weak noncompactness, and certain spectral inequalities are formulated for Lions–Schechter's complex methods of interpolation with derivatives.

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The theory of the complex interpolation methods owes its origin to the famous interpolation theorem of Riesz–Thorin [3, Theorem 1.1.1]. Some fundamental inequalities due to Calderón [3, Lemma 4.3.2] imply that the boundedness of linear operators can be interpolated between Banach spaces with a logarithmically convex estimate for the norms of the interpolated operators. This motivated several authors to investigate the behavior of other properties of Banach spaces and linear operators under interpolation in both qualitative and quantitative way. For instance, Salvatori and Vignati obtained the estimate for multi-dimensional rotundity [11], Kryczka and Prus studied the measure of weak noncompactness [9], and Albrecht and Müller formulated some spectral inequalities under the complex interpolation methods [2].

In the present paper, we treat the similar problems for the more general Lions–Schechter's methods of complex interpolation with derivatives. In the first section, we review the different variants of these methods and formulate some basic inequalities. Section 2 includes the estimate

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for multi-dimensional rotundity of Banach spaces. Section 3 is devoted to the measure of weak noncompactness and spectral inequalities for bounded linear operators.

We use standard interpolation theory notation as can be found in [3]. Let $\overline{X} = (X_0, X_1)$ be a Banach couple with $\Delta\overline{X} = X_0 \cap X_1$ and $\Sigma\overline{X} = X_0 + X_1$, and let X be an intermediate space for Banach couple $\overline{X} = (X_0, X_1)$, we denote by X^0 the regularization for $\overline{X}$, by X' the Banach space dual of X^0, and write the dual couple $\overline{X}' = (X_0', X_1')$. For two Banach spaces X and Y, we denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators from X to Y equipped with the operator norm. For two Banach couples $\overline{X}$ and $\overline{Y}$, we denote by $\mathcal{B}(\overline{X}, \overline{Y})$ the Banach space of all bounded linear operators T from $\overline{X}$ to $\overline{Y}$, for which $T \in \mathcal{B}(X_j, Y_j)$ with the norm $\|T\|_j$ $(j = 0, 1)$, and

$$\|T\|_{\overline{X},\overline{Y}} = \|T\|_0 \vee \|T\|_1.$$

Here $\alpha \vee \beta = \max\{\alpha, \beta\}$ for $\alpha, \beta \in \mathbf{R}$. We simply write $\mathcal{B}(X) = \mathcal{B}(X, X)$ and $\mathcal{B}(\overline{X}) = \mathcal{B}(\overline{X}, \overline{X})$. Throughout the paper, the notations $\subseteq$ and $=$ between Banach spaces stand for continuous inclusion and isomorphic equivalence, respectively.

1. On variants of Lions–Schechter's interpolation methods

For a Banach couple $\overline{X}$ on the strip $\mathbf{S} = \{z \in \mathbf{C} \mid 0 \leqslant \operatorname{Re} z \leqslant 1\}$, let $\mathcal{A}^b(\mathbf{S}, \overline{X})$ be the Banach space of all continuous functions $f : \mathbf{S} \longrightarrow \Sigma\overline{X}$, where f is analytic in the interior of $\mathbf{S}$ and, for $j = 0, 1$, the function $t \mapsto f(j + it)$ is boundedly continuous from $\mathbf{R}$ to X_j. This space is equipped with the norm

$$\|f\|_\infty = \max_{j=0,1} \left\{ \sup_{t\in\mathbf{R}} \|f(j + it)\|_j \right\}$$

for $f \in \mathcal{A}^b(\mathbf{S}, \overline{X})$. The complex interpolation spaces with nth derivative at θ, where $n \in \mathbf{N}$ and $0 < \theta < 1$, is defined in the following way:

$$C_{\theta(n)}(\overline{X}) = \left\{ x \in \Sigma\overline{X} \,\middle|\, x = \frac{1}{n!}\left(-\frac{2\sin\pi\theta}{\pi}\right)^n f^{(n)}(\theta), \;\; f \in \mathcal{A}^b(\mathbf{S}, \overline{X}) \right\}$$

with the norm $\|x\|_{C_{\theta(n)}} = \inf\left\{ \|f\|_\infty \,\middle|\, x = \frac{1}{n!}\left(-\frac{2\sin\pi\theta}{\pi}\right)^n f^{(n)}(\theta) \right\}$, and

$$C_{\theta(-n)}(\overline{X}) = \left\{ x \in \Sigma\overline{X} \,\middle|\, x = f(\theta), \;\; f \in \mathcal{A}^b(\mathbf{S}, \overline{X}), \;\; f^{(k)}(\theta) = 0, \;\; 1 \leqslant k \leqslant n \right\}$$

with the norm $\|x\|_{C_{\theta(-n)}} = \inf\left\{ \|f\|_\infty \,\middle|\, x = f(\theta) \right\}$. In particular, we set

$$C_\theta(\overline{X}) = C_{\theta(0)}(\overline{X}).$$

For $0 < \theta < 1$ and $n \geqslant 1$, we define submultiplicative functions $\rho_{\theta,n}$ on $\mathbf{R}^+$ by

$$\rho_{\theta,1}(t) = t^\theta \left(\sqrt{1 + \left(\frac{\sin\pi\theta}{\pi}|\log t|\right)^2} + \frac{\sin\pi\theta}{\pi}|\log t| \right)$$

and

$$\rho_{\theta,n}(t) = t^\theta \left(1 + \frac{\theta(1-\theta)}{n} |\log t|\right)^n \quad \text{if } n \geqslant 2.$$

In addition, we set

$$\rho_{\theta,n}(0) = \lim_{t\to 0+} \rho_{\theta,n}(t) = 0.$$

The corresponding homogeneous function of two variables (again denoted by $\rho_{\theta,n}$) is given by $(t_0, t_1) \longmapsto t_0 \rho_{\theta,n}(t_1/t_0)$ for $t_0,\ t_1 > 0$. It is clear

$$\rho_{\theta,1}\big(t_0, t_1\big) = t_0^{1-\theta} t_1^\theta \left(\sqrt{1 + \left(\frac{\sin \pi\theta}{\pi}\left|\log \frac{t_1}{t_0}\right|\right)^2} + \frac{\sin \pi\theta}{\pi}\left|\log \frac{t_1}{t_0}\right|\right),$$

and

$$\rho_{\theta,n}\big(t_0, t_1\big) = t_0^{1-\theta} t_1^\theta \left(1 + \frac{\theta(1-\theta)}{n}\left|\log \frac{t_1}{t_0}\right|\right)^n \quad \text{if } n \geqslant 2.$$

The $C_{\theta(\pm n)}$-methods are of interpolation type $\rho_{\theta,n}$ in the sense that the inequality

$$\|T\|_{C_{\theta(\pm n)}(\overline{X}), C_{\theta(\pm n)}(\overline{Y})} \leqslant c\, \rho_{\theta,n}\Big(\|T\|_0, \|T\|_1\Big) \tag{1.1}$$

holds for all Banach couples $\overline{X}$, $\overline{Y}$, and for all operators $T \in \mathcal{B}\big(\overline{X}, \overline{Y}\big)$, where c is a constant only depending on θ and n. Specially, $c = 1$ for $n = 1$. Furthermore, for $0 < \theta_0, \theta_1, \zeta < 1$ with $\theta_0 < \theta < \theta_1$, $\theta = (1-\zeta)\theta_0 + \zeta\theta_1$ and for $k = 1, 2, \ldots,$ we have reiterations

$$C_{\theta(\pm n)}\big(\overline{X}\big) = C_{\zeta(\pm n)}\Big(C_{\theta_0}\big(\overline{X}\big), C_{\theta_1}\big(\overline{X}\big)\Big) \tag{1.2}$$

by [7, Remark 5.8], and

$$C_{\zeta(-n)}\Big(C_{\theta_0(-k)}\big(\overline{X}\big) C_{\theta_1(-k)}\big(\overline{X}\big)\Big) \subseteq C_{\theta(-n-k)}\big(\overline{X}\big) \tag{1.3}$$

by [5, Theorem 5.1].

Some variants of the $C_{\theta(n)}$-methods on the unit disk $\mathbf{D} = \big\{z \in \mathbf{C} \,\big|\, |z| \leqslant 1\big\}$ are systematically developed in [7] and [6]. Let

$$I_0^\theta = \big[-\pi(1-\theta), \pi(1-\theta)\big), \quad I_1^\theta = \big[\pi(1-\theta), \pi(1+\theta)\big).$$

We denote by $A_\theta\big(\mathbf{D}, \overline{X}\big)$ the Banach space of all functions $f\colon \mathbf{D} \longrightarrow \Sigma\overline{X}$, where f is analytic in the interior of $\mathbf{D}$ and, for $j = 0, 1$, the function $t \mapsto f(e^{it})$ is continuous from I_j^θ to X_j, with the norm

$$\|f\|_{\mathcal{A}_\theta} = \max_{j=0,1} \sup\Big\{ \|f(e^{it})\|_j \,\Big|\, t \in I_j^\theta \Big\}.$$

Let $P\big(\mathbf{D}, \Delta\overline{X}\big)$ be the set of all polynomials on $\mathbf{D}$ with coefficients in $\Delta\overline{X}$. We denote by $\mathcal{H}_\theta^1\big(\mathbf{D}, \overline{X}\big)$ the Banach space completion of $P\big(\mathbf{D}, \Delta\overline{X}\big)$ with the norm

$$\|f\|_{\mathcal{H}_\theta^1} = \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \left\|f\big(e^{it}\big)\right\|_0 dt\right) \vee \left(\frac{1}{2\pi\theta} \int_{I_1^\theta} \left\|f\big(e^{it}\big)\right\|_1 dt\right).$$

We mention here that the corresponding spaces in [6] should also be defined in this way. Now we introduce

$$C_{D\theta(n,\infty)}\big(\overline{X}\big) = \left\{ x \in \Sigma\overline{X} \;\middle|\; x = \hat{f}(n) = \frac{1}{n!} f^{(n)}(0), \quad f \in \mathcal{A}_\theta\big(\mathbf{D}, \overline{X}\big) \right\}$$

with the norm $\|x\|_{C_{D\theta(n,\infty)}} = \inf\Big\{ \|f\|_{\mathcal{A}_\theta} \;\Big|\; x = \hat{f}(n) \Big\}$, and

$$C_{D\theta(-n,\infty)}\big(\overline{X}\big) = \left\{ x \in \Sigma\overline{X} \;\middle|\; x = f(0), \;\; f \in \mathcal{A}_\theta\big(\mathbf{D}, \overline{X}\big), \;\; \hat{f}(l) = 0, \; 1 \leqslant l \leqslant n \right\}$$

with the norm $\|x\|_{C_{D\theta(-n,\infty)}} = \inf\Big\{ \|f\|_{\mathcal{A}_\theta} \;\Big|\; x = f(0) \Big\}$. Similarly, the spaces $C_{D\theta(\pm n,1)}\big(\overline{X}\big)$ are defined by replacing $\mathcal{A}_\theta\big(\mathbf{D}, \overline{X}\big)$ with $\mathcal{H}^1_\theta\big(\mathbf{D}, \overline{X}\big)$. The equivalence

$$C_{\theta(n)}\big(\overline{X}\big) = C_{D\theta(n,1)}\big(\overline{X}\big) = C_{D\theta(n,\infty)}\big(\overline{X}\big) \tag{1.4}$$

for $n \in \mathbf{Z}$ follows by the conformal mapping $m_\theta \colon \mathbf{S} \longrightarrow \mathbf{D}$, for which

$$m_\theta(z) = \frac{\exp\big(i\pi(z-\theta)\big) - 1}{\exp(-i\pi\theta) - \exp(i\pi z)} \quad \text{for } z \in \mathbf{S}.$$

For $n = 0$, the equivalence in (1.4) is isometric. For $n = 1$, we have the norm estimate

$$\|x\|_{C_{D\theta(\pm 1,1)}} = \|x\|_{C_{\theta(\pm 1)}} \leqslant \|x\|_{C_{D\theta(\pm 1,\infty)}} \leqslant \frac{3\sqrt{3}}{4} \|x\|_{C_{D\theta(\pm 1,1)}}$$

by [7, Theorem 5.2]. For $n > 1$, the equivalence constants appearing in (1.4) only depend on θ and n. Moreover, if X_0 or X_1 is reflexive, then the duality

$$C_{D\theta(\pm n,1)}\big(\overline{X}\big)' = C_{D\theta(\mp n,\infty)}\big(\overline{X}'\big) \tag{1.5}$$

holds true with the isometric norms.

We conclude this section by some basic inequalities for the $C_{D\theta(\pm n,1)}$ methods, which will play a key role in the proof of our results concerning various quantitative estimates under Lions–Schechter's complex interpolation. Let $f \in \mathcal{H}^1_\theta\big(\mathbf{D}, \overline{X}\big)$. Then there exists $h \in \mathcal{H}^1_\theta\big(\mathbf{D}, \overline{X}\big)$ with $\hat{h}(l) = \hat{f}(l)$, for $0 \leqslant l \leqslant n$, such that

$$\|h\|_{\mathcal{H}^1_\theta} \leqslant \rho_{\theta,1} \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \left\| f\big(e^{it}\big) \right\|_0 dt, \; \frac{1}{2\pi\theta} \int_{I_1^\theta} \left\| f\big(e^{it}\big) \right\|_1 dt \right), \tag{1.6}$$

and

$$\|h\|_{\mathcal{H}^1_\theta} \leqslant 3^n \rho_{\theta,n} \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \left\| f\big(e^{it}\big) \right\|_0 dt, \; \frac{1}{2\pi\theta} \int_{I_1^\theta} \left\| f\big(e^{it}\big) \right\|_1 dt \right) \tag{1.7}$$

for $n \geqslant 2$. We refer to [6, Lemma 2.1] for the proof of these inequalities.

2. On multi-dimensional rotundity of Banach spaces

Let X be a Banach space, and let $x_v \in X$ for $0 \leqslant v \leqslant k$. The k-dimensional volume enclosed by $x_0, x_1, \ldots, x_k$ is described by

$$V_X\big(\{x_v\}\big) = \sup_{x_v^* \in X', \|x_v^*\|_{X'} \leqslant 1} \left\{ \left| \begin{matrix} 1 & \cdots & 1 \\ \langle x_1^*, x_0\rangle & \cdots & \langle x_1^*, x_k\rangle \\ \vdots & \ddots & \vdots \\ \langle x_k^*, x_0\rangle & \cdots & \langle x_k^*, x_k\rangle \end{matrix} \right| \right\}.$$

The modulus of k-rotundity of X is defined as follows:

$$\delta_X^{(k)}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x_0 + x_1 + \cdots + x_k}{k+1} \right\|_X \,\middle|\, \|x_v\|_X \leqslant 1,\ V_X\big(\{x_v\}\big) \geqslant \varepsilon \right\}$$

for $0 \leqslant \varepsilon \leqslant (k+1)^{(k+1)/2}$. The space X is k-uniformly rotund (k-UR in short), or equivalently k-uniformly convex, if $\delta_X^{(k)}(\varepsilon) > 0$ for $\varepsilon > 0$. The characteristic of k-convexity of X is defined by

$$\hat{\varepsilon}_X^{(k)} = \inf\big\{ \varepsilon \,\big|\, \delta_X^{(k)}(\varepsilon) > 0 \big\}.$$

Lemma 2.1. *Assume that X_0 or X_1 is reflexive. Let $X = C_{D\theta(1,1)}\big(\overline{X}\big)$, and let $x_0, \ldots, x_k \in X$. If $x_v = f_v'(0)$ for some $f_v \in \mathcal{H}_\theta^1\big(\mathbf{D}, \overline{X}\big)$, $0 \leqslant v \leqslant k$, then*

$$V_X\big(\{x_v\}\big) \leqslant \rho_{1,\theta}\left(\left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} V_{X_0}\Big(\big\{f_v(e^{it})\big\}\Big)\, dt \right)^{1/k}, \left(\frac{1}{2\pi\theta} \int_{I_1^\theta} V_{X_1}\Big(\big\{f_v(e^{it})\big\}\Big)\, dt \right)^{1/k} \right)^k.$$

Proof. Let $\eta > 0$ fixed. By duality (1.5), we may choose $g_m \in \mathcal{A}_\theta\big(\mathbf{D}, \overline{X}'\big)$, $1 \leqslant m \leqslant k$, with norms $\leqslant 1$ satisfying $x_m^* = (1+\eta) g_m(0)$, $g_m'(0) = 0$, and

$$V_X\big(\{x_v\}\big) \leqslant V\big(\{x_v, x_m^*\}\big),$$

where

$$V\big(\{x_v, x_m^*\}\big) = \left| \begin{matrix} 1 & \cdots & 1 \\ \langle x_1^*, x_0\rangle & \cdots & \langle x_1^*, x_k\rangle \\ \vdots & \ddots & \vdots \\ \langle x_k^*, x_0\rangle & \cdots & \langle x_k^*, x_k\rangle \end{matrix} \right|.$$

Let now

$$V\big(\cdot\,; \{f_v, g_m\}\big) = \left| \begin{matrix} 1 & \cdots & 1 \\ \langle g_1, f_0\rangle & \cdots & \langle g_1, f_k\rangle \\ \vdots & \ddots & \vdots \\ \langle g_k, f_0\rangle & \cdots & \langle g_k, f_k\rangle \end{matrix} \right|.$$

Then $V\big(\cdot\,;\{f_\nu, g_m\}\big) \in \mathcal{H}^1(\mathbf{D})$ with

$$V\big(\{x_\nu, x_m^*\}\big) = (1+\eta)^k V\big(0;\{f'_\nu, g_m\}\big) = (1+\eta)^k V'\big(0;\{f_\nu, g_m\}\big),$$

which implies that

$$V\big(\{x_\nu, x_m^*\}\big) = (1+\eta)^k \int_0^{2\pi} e^{-it} V\big(e^{it};\{f_\nu, g_m\}\big)\,dt.$$

For $t \in I_j^\theta$ $(j=0,1)$, let $\varepsilon_j(t) = V_{X_j}\Big(\big\{f_\nu(e^{it})\big\}\Big)$, and let

$$M_0 = \left(\frac{1}{2\pi(1-\theta)}\int_{I_0^\theta} \varepsilon_0(t)\,dt\right)^{1/k} \quad \text{and} \quad M_1 = \left(\frac{1}{2\pi\theta}\int_{I_1^\theta} \varepsilon_1(t)\,dt\right)^{1/k}.$$

As in the proof of [6, Lemma 2.1], there exist $\phi, \psi \in \mathcal{H}^\infty(\mathbf{D})$ such that

$$\phi(0) = 0, \quad \phi'(0) = \frac{\sin\pi\theta}{\pi}\log\big(M_1/M_0\big),$$
$$\|\phi\|_\infty = \sqrt{1+\left(\frac{\sin\pi\theta}{\pi}\left|\log\frac{M_1}{M_0}\right|\right)^2} + \frac{\sin\pi\theta}{\pi}\left|\log\frac{M_1}{M_0}\right|,$$

and

$$\Big|\exp\big(-\psi(e^{it})\big)\Big| = \left(\frac{M_0}{M_1}\right)^{j-\theta} \qquad \text{for } t \in I_j^\theta \ \ (j=0,1).$$

Furthermore, let $h_m = \phi\cdot\exp(-\psi)\cdot g_m$, $1 \leqslant m \leqslant k$. Then $h_m \in \mathcal{H}_\theta^1\big(\mathbf{D}, \overline{X}\big)'$ with $(1+\eta)h_m(0) = (1+\eta)g_m(0) = x_m^*$, $h'_m(0) = 0$ and

$$\sup_{t\in I_j^\theta}\big|h_m(e^{it})\big| \leqslant M_j^{-1}\rho_{\theta,1}\big(M_0, M_1\big) \quad (j=0,1).$$

Consequently,

$$V\big(\{x_\nu, x_m^*\}\big) = (1+\eta)^k \int_0^{2\pi} e^{-it} V\big(e^{it};\{f_\nu, h_m\}\big)\,dt,$$

and hence

$$V_X\big(\{x_\nu\}\big) \leqslant V\big(\{x_\nu, x_m^*\}\big) \leqslant (1+\eta)^k \rho_{\theta,1}\big(M_0, M_1\big)^k.$$

We complete the proof by letting $\eta \to 0$. □

Lemma 2.2. *Let φ and ψ be concave and increasing functions defined on* $[0,1)$ *with* $\varphi(0) = \psi(0) = 0$. *If* $s,\ t,\ u \in (0,1)$ *with* $u \leqslant \rho_{\theta,1}\big(s^{1/k}, t^{1/k}\big)^k$, *then*

$$\rho_{\theta,1}\big(\varphi(1-s)^{1/k}, \psi(1-t)^{1/k}\big) \leqslant c^{1/k}\,\rho_{\theta,1}\big(\varphi(1-u)^{1/k}, \psi(1-u)^{1/k}\big),$$

where

$$c=\frac{1}{1-\theta_1}\left(\left(\frac{1-\theta_1}{\theta_0}\right)^{\theta_0}\vee\left(\frac{1-\theta_1}{\theta_0}\right)^{\theta_1}\right)$$

for $\theta_0=\theta-\sin\pi\theta/\pi$ *and* $\theta_1=\theta+\sin\pi\theta/\pi$.

Proof. It is clear that $0<\theta_0<\theta_1<1$. Observe that, for $s,\ t>0$,

$$\rho_{\theta,1}(st)\leqslant\left(s^{\theta_0}\vee s^{\theta_1}\right)\rho_{\theta,1}(t).$$

If $s\leqslant t$, then

$$u\leqslant s\rho_{\theta,1}\left(\left((t/s)^{1/k}\right)^k\right)\leqslant s^{1-\theta_1}t^{\theta_1}\leqslant(1-\theta_1)s+\theta_1 t,$$

and hence $1-u\geqslant(1-\theta_1)(1-s)+\theta_1(1-t)$. This implies that

$$\begin{aligned}\varphi(1-u)&\geqslant(1-\theta_1)\varphi(1-s)+\theta_1\varphi(1-t)\geqslant(1-\theta_1)\varphi(1-s),\\ \psi(1-u)&\geqslant(1-\theta_1)\psi(1-s)+\theta_1\psi(1-t)\geqslant\theta_1\psi(1-t)\geqslant\theta_0\psi(1-t).\end{aligned}$$

Similarly, if $s\geqslant t$, then we have

$$\varphi(1-u)\geqslant(1-\theta_0)\varphi(1-s)\geqslant(1-\theta_1)\varphi(1-s)\quad\text{and}\quad\psi(1-u)\geqslant\theta_0\psi(1-t).$$

Therefore,

$$\begin{aligned}&\rho_{\theta,1}\left(\varphi(1-s)^{1/k},\psi(1-t)^{1/k}\right)\\ &\quad\leqslant\rho_{\theta,1}\left(\left(\varphi(1-u)/(1-\theta_1)\right)^{1/k},\left(\psi(1-u)/\theta_0\right)^{1/k}\right)\\ &\quad\leqslant c^{1/k}\,\rho_{\theta,1}\left(\varphi(1-u)^{1/k},\psi(1-u)^{1/k}\right),\end{aligned}$$

which completes the proof. □

Given two nonnegative functions f, g defined on an interval $[0,c)$, we denote $f\succ g$ (or $g\prec f$) if there exist positive constants α,β such that $f(t)\geqslant\alpha g(\beta t)$ for t small enough. We now state the main result of this section.

Proposition 2.1. *If X_0 or X_1 is k-UR, then $X=C_{D\theta(\pm1,1)}(\overline{X})$ is also k-UR. Moreover, if we set $\delta_j=\delta_{X_j}^{(k)}$ be the modulus of k-rotundity of X_j $(j=0,1)$, then the modulus of k-rotundity $\delta_X=\delta_X^{(k)}$ of X satisfies*

(i) $\delta_X(\varepsilon)\succ1-\left(1-\delta_0\left(\varepsilon^{1/(1-\theta_1)}\right)\right)^{1-\theta_1}$ *when X_0 is k-UR*;

(ii) $\delta_X\succ\left(\rho_{\theta,1}\left(\left(\delta_0^{-1}\right)^{1/k},\left(\delta_1^{-1}\right)^{1/k}\right)^k\right)^{-1}$ *when both X_0 and X_1 are k-UR.*

Proof. We prove the proposition for the space $X=C_{D\theta(1,1)}(\overline{X})$. The other case can be treated in an analogous way. By [11, Proposition 1], we may assume that δ_j is a convex function and is strictly increasing when X_j is k-UR. Let $0<\varepsilon<(k+1)^{(k+1)/2}$, and let $x_0,\dots,x_k\in X$ with $\|x_v\|_{C_{D\theta(1,1)}}<1$ and $V_X(\{x_v\})\geqslant\varepsilon$. For $\eta>0$ fixed and for each $v=0,\dots,k$, there exists a function $f_v \in \mathcal{H}^1(\mathbf{D}_\theta, \overline{X})$ with $\|f\|_{\mathcal{H}^1} \leqslant 1$ and $(1+\eta) f_v'(0) = x_v$. Let now

$$\sigma(z) = \frac{1}{k+1} \sum_{v=0}^{k} f_v(z)$$

and $\varepsilon_j(t) = V_{X_j}\Big(\big\{f_v(e^{it})\big\}\Big)$ for $t \in I_j^\theta$ $(j = 0, 1)$. Then

$$\delta_j\big(\varepsilon_j(t)\big) \leqslant 1 - \big\|\sigma(e^{it})\big\|_j \tag{2.1}$$

for $t \in I_j^\theta$ $(j = 0, 1)$. By (1.6) and Lemma 2.1, we have

$$\begin{aligned} &\frac{1}{1+\eta} \left\| \frac{1}{k+1} \sum_{v=0}^{k} x_v \right\|_X \\ &\quad \leqslant \rho_{\theta,1} \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \big\|\sigma(e^{it})\big\|_0 \, dt, \frac{1}{2\pi\theta} \int_{I_1^\theta} \big\|\sigma(e^{it})\big\|_1 \, dt \right), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\frac{\varepsilon}{(1+\eta)^k} \leqslant V_X\Big(\big\{f_v'(0)\big\}\Big) \\ &\quad \leqslant \rho_{\theta,1} \left(\left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \varepsilon_0(t) \, dt \right)^{1/k}, \left(\frac{1}{2\pi\theta} \int_{I_1^\theta} \varepsilon_1(t) \, dt \right)^{1/k} \right)^k. \end{aligned} \tag{2.3}$$

(i) If X_0 is k-UR, then by (2.3) and by the estimate

$$\varepsilon_1(t) = V_{X_1}\Big(\big\{f_v(e^{it})\big\}\Big) \leqslant (k+1)^{(k+1)/2} = a_k \quad \text{for } t \in I_1^\theta,$$

we have

$$\begin{aligned} \frac{\varepsilon}{(1+\eta)^k} &\leqslant \rho_{\theta,1} \left(\left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \varepsilon_0(t) \, dt \right)^{1/k}, a_k^{1/k} \right)^k \\ &\leqslant a_k^{\theta_1} \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \varepsilon_0(t) \, dt \right)^{1-\theta_1}. \end{aligned}$$

This, combined with (2.1) and (2.2), yields that

$$\begin{aligned} \left\| \frac{1}{k+1} \sum_{v=0}^{k} x_v \right\|_X &\leqslant (1+\eta)\rho_{\theta,1} \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \big\|\sigma(e^{it})\big\|_0 \, dt, 1 \right) \\ &\leqslant (1+\eta)\rho_{\theta,1} \left(1 - \delta_0 \left(\frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \varepsilon_0(t) \, dt \right), 1 \right) \\ &\leqslant (1+\eta)\rho_{\theta,1}\Big(1 - \delta_0\Big(\big(a_k^{-\theta_1}(1+\eta)^{-k}\varepsilon\big)^{1/(1-\theta_1)}\Big), 1\Big) \\ &\leqslant (1+\eta)\Big(1 - \delta_0\Big(\big(a_k^{-\theta_1}(1+\eta)^{-k}\big)\varepsilon^{1/(1-\theta_1)}\Big)\Big)^{1-\theta_1}. \end{aligned}$$

Consequently,

$$1-\left(1-\delta_0\left(\left(a_k^{-\theta_1}\varepsilon\right)^{1/(1-\theta_1)}\right)\right)^{1-\theta_1}\leqslant 1-\left\|\frac{1}{k+1}\sum_{v=0}^{k}x_v\right\|_X$$

by letting $\eta\to 0$, and hence $\delta_X(\varepsilon)\succ 1-\left(1-\delta_0\left(\varepsilon^{1/(1-\theta_1)}\right)\right)^{1-\theta_1}$.

(ii) If both X_0 and X_1 are k-UR, then both δ_0^{-1} and δ_1^{-1} are concave functions. Let us now choose

$$\varphi=\delta_0^{-1},\quad \psi=\delta_1^{-1},\quad s=\frac{1}{2\pi(1-\theta)}\int_{I_0^\theta}\left\|\sigma(e^{it})\right\|_0dt,$$

$$t=\frac{1}{2\pi\theta}\int_{I_1^\theta}\left\|\sigma(e^{it})\right\|_1dt\quad\text{and}\quad u=\frac{1}{1+\eta}\left\|\frac{1}{k+1}\sum_{v=0}^{k}x_v\right\|_X.$$

Observe that

$$\frac{\varepsilon}{(1+\eta)^k}\leqslant\rho_{\theta,1}\left(\left(\frac{1}{2\pi(1-\theta)}\int_{I_0^\theta}\delta_0^{-1}\left(1-\left\|\sigma(e^{it})\right\|_0\right)dt\right)^{1/k},\right.$$
$$\left.\left(\frac{1}{2\pi\theta}\int_{I_1^\theta}\delta_1^{-1}\left(1-\left\|\sigma(e^{it})\right\|_1\right)dt\right)^{1/k}\right)^k$$

by (2.1) and (2.3). This, together with Jensen's inequality, Lemma 2.2 and (2.2), yields that

$$\frac{\varepsilon}{(1+\eta)^k}\leqslant\rho_{\theta,1}\left(\varphi(1-s)^{1/k},\psi(1-t)^{1/k}\right)^k\leqslant c\,\rho_{\theta,1}\left(\varphi(1-u)^{1/k},\psi(1-u)^{1/k}\right)^k.$$

That is,

$$\varepsilon\leqslant c\,(1+\eta)^k\rho_{\theta,1}\left(\delta_0^{-1}\left(1-\frac{1}{1+\eta}\left\|\frac{1}{k+1}\sum_{v=0}^{k}x_v\right\|_X\right)^{1/k},\right.$$
$$\left.\delta_1^{-1}\left(1-\frac{1}{1+\eta}\left\|\frac{1}{k+1}\sum_{v=0}^{k}x_v\right\|_X\right)^{1/k}\right)^k.$$

It gives that $\varepsilon\leqslant c\,\rho_{\theta,1}\left(\delta_0^{-1}\left(\delta_X(\varepsilon)\right)^{1/k},\delta_1^{-1}\left(\delta_X(\varepsilon)\right)^{1/k}\right)^k$, and hence

$$\delta_X^{-1}(\varepsilon)\leqslant c\,\rho_{\theta,1}\left(\delta_0^{-1}(\varepsilon)^{1/k},\delta_1^{-1}(\varepsilon)^{1/k}\right)^k.$$

Therefore, we obtain $\delta_X\succ\left(\rho_{\theta,1}\left((\delta_0^{-1})^{1/k},(\delta_1^{-1})^{1/k}\right)^k\right)^{-1}$. □

In the following example, we estimate the modulus of k-rotundity of some concrete function spaces.

Example. Let (Ω, μ) be a complete σ-finite measure space such that $\mathcal{L}^2 = \mathcal{L}^2(\Omega, \mu)$ is an infinite dimensional Hilbert space. Then, by [10, Example 3.13],

$$\delta^{(k)}_{\mathcal{L}^2}(\varepsilon) = 1 - \left(1 - \frac{k}{(k+1)^{1+1/k}}\varepsilon^{2/k}\right)^{1/2} \geqslant \frac{k}{(k+1)^{1+1/k}}\varepsilon^{2/k}.$$

Recall that, for $p > 2$ and $\theta = 1 - 2/p$,

$$\mathcal{L}^p = \mathcal{L}^p(\Omega, \mu) = C_\theta\big(\mathcal{L}^2(\Omega, \mu), \mathcal{L}^\infty(\Omega, \mu)\big)$$

isometrically. This implies that

$$\delta^{(k)}_{\mathcal{L}^p}(\varepsilon) \succ \delta^{(k)}_{\mathcal{L}^2}\big(\varepsilon^{1/(1-\theta)}\big) = \delta^{(k)}_{\mathcal{L}^2}\big(\varepsilon^{p/2}\big) \succ \varepsilon^{p/k}$$

by [11, Theorem 2]. Let now $2 \leqslant p_0 < p_1 < \infty$ with $1/q = 1/p_0 - 1/p_1$, and let $\varphi_{\theta,1}$ be the function defined on $\mathbf{R}^+$ by

$$\varphi^{-1}_{\theta,1}(t) = t^{1/p_0}\rho_{\theta,1}\big(t^{-1/q}\big).$$

If we denote by $\mathcal{L}^{\varphi_{\theta,1}} = \mathcal{L}^{\varphi_{\theta,1}}(\Omega, \mu)$ the corresponding Orlicz space over (Ω, μ) equipped with the Luxemberg norm, then by [6, Remark 4.8], we have

$$\mathcal{L}^{\varphi_{\theta,1}}(\Omega, \mu) = C_{D\theta(1,1)}\big(\mathcal{L}^{p_0}(\Omega, \mu), \mathcal{L}^{p_1}(\Omega, \mu)\big)$$

isomorphically. We may apply Proposition 2.1 on

$$X = C_{D\theta(1,1)}\big(\mathcal{L}^{p_0}(\Omega, \mu), \mathcal{L}^{p_1}(\Omega, \mu)\big),$$

and hence obtain

$$\begin{aligned}\big(\delta^{(k)}_X(\varepsilon)\big)^{-1} &\prec \rho_{\theta,1}\Big(\big(\delta^{(k)}_{\mathcal{L}^{p_0}}\big)^{-1}(\varepsilon)^{1/k}, \big(\delta^{(k)}_{\mathcal{L}^{p_1}}\big)^{-1}(\varepsilon)^{1/k}\Big)^k \\ &\prec \rho_{\theta,1}\big(\varepsilon^{1/p_0}, \varepsilon^{1/p_1}\big)^k = \varphi^{-1}_{\theta,1}(\varepsilon)^k.\end{aligned}$$

Therefore, $\delta^{(k)}_X(\varepsilon) \succ \varphi_{\theta,1}\big(\varepsilon^{1/k}\big)$.

Let $0 < \theta_0, \theta_1, \zeta < 1$ with $\theta_0 < \theta < \theta_1$ and $\theta = (1-\zeta)\theta_0 + \zeta\theta_1$. If we apply Proposition 2.1 on the space

$$C_{\zeta(-n)}\Big(C_{\theta_0(-1)}\big(\overline{X}\big), C_{\theta_1(-1)}\big(\overline{X}\big)\Big)$$

inductively, then together with reiteration (1.3), we have

Proposition 2.2. *If X_0 or X_1 is k-UR and if $n \geqslant 1$, then $X = C_{\theta(-n)}\big(\overline{X}\big)$ is equivalent to a k-UR space.*

By a slight modification on the proof of [4, Theorem 2.1], we obtain the following result concerning the characteristic of *k*-convexity.

Proposition 2.3. *For $0 < \theta < 1$, let $\hat{\varepsilon}^{(k)}_\theta$ be the characteristic of k-convexity of the complex interpolation space $C_\theta\big(\overline{X}\big)$. Then*

$$\hat{\varepsilon}^{(k)}_\theta \leqslant \big(\hat{\varepsilon}^{(k)}_{X_0}\big)^{1-\theta}\big(\hat{\varepsilon}^{(k)}_{X_1}\big)^\theta.$$

3. On measure of weak noncompactness and some spectral inequalities for operators

In this section, let us first consider the measure of weak noncompactness for bounded linear operators given by Kryczka and Prus [9], and apply their idea on the $C_{\theta(\pm n)}$-methods. Let X be a Banach space and let $(x_v)_{v\geqslant 1}$ be a sequence in X. Vectors u_1, u_2 in X are said to be a pair of successive convex combinations (scc for short) for $(x_v)_v$ if $u_1 \in \operatorname{conv}\left(\{x_v\}_{v=1}^k\right)$ and $u_2 \in \operatorname{conv}\left(\{x_v\}_{v=k+1}^\infty\right)$ for some integer $k\geqslant 1$. By the convex separation of $(x_v)_v$, denoted by $\operatorname{csep}\left((x_v)_v\right)$, we mean

$$\operatorname{csep}\left((x_v)_v\right) = \inf\left\{ \|u_1 - u_2\|_X \,\middle|\, u_1,\ u_2 \text{ is a pair of scc for } (x_v)_v \right\}.$$

For a nonempty and bounded subset A of X, we define the measure of weak noncompactness of A by

$$\gamma(A) = \sup\left\{ \operatorname{csep}\left((x_v)_v\right) \,\middle|\, (x_v)_v \subseteq \operatorname{conv} A \right\}.$$

For Banach spaces X and Y, let U_X be the open unit ball of X and let $T \in \mathcal{B}(X, Y)$. We define the measure of weak noncompactness of operator T by

$$\Gamma(T) = \gamma\big(T(U_X)\big).$$

Proposition 3.1. *Let $T \in \mathcal{B}\big(\overline{X}, \overline{Y}\big)$. Then*

$$\Gamma_{\theta(\pm n)}(T) \leqslant c\, \rho_{\theta,n}\big(\Gamma_0(T), \Gamma_1(T)\big),$$

where $\Gamma_{\theta(\pm n)}(T)$ and $\Gamma_j(T)$ $(j = 0, 1)$ are measures of weak noncompactness for operators $T: C_{\theta(\pm n)}\big(\overline{X}\big) \to C_{\theta(\pm n)}\big(\overline{Y}\big)$ and $T: X_j \to Y_j$ $(j = 0, 1)$, respectively; and c is a constant only depending on θ and n. Consequently, if $T: X_0 \to Y_0$ or $T: X_1 \to Y_1$ is weakly compact, then

$$T: C_{\theta(\pm n)}\big(\overline{X}\big) \to C_{\theta(\pm n)}\big(\overline{Y}\big)$$

is also weakly compact.

Proof. By equivalence (1.4), we consider spaces

$$X = C_{D\theta(n,\infty)}\big(\overline{X}\big) \quad \text{and} \quad Y = C_{D\theta(n,1)}\big(\overline{Y}\big).$$

For $j = 0, 1$, let $\mathcal{C}_j[X_j]$ be the space of all continuous functions $f: I_j^\theta \to X_j$ with the norm $\|f\|_{\mathcal{C}_j} = \max_{t\in I_j^\theta} |f(t)|$, and let $\mathcal{L}_j^1[X_j]$ be the space of all measurable functions $f: I_j^\theta \to X_j$ such that

$$\|f\|_{\mathcal{L}_0^1} = \frac{1}{2\pi(1-\theta)} \int_{I_0^\theta} \|f(t)\|_0\, dt < \infty \quad \text{for } f \in \mathcal{L}_0^1[X_0]$$

and

$$\|f\|_{\mathcal{L}_1^1} = \frac{1}{2\pi\theta} \int_{I_1^\theta} \|f(t)\|_1\, dt < \infty \quad \text{for } f \in \mathcal{L}_1^1[X_1].$$

The operator $T \in \mathcal{B}(\overline{X}, \overline{Y})$ induces operators $\widetilde{T}_j \in \mathcal{B}(\mathcal{C}_j[X_j], \mathcal{L}^1_j[Y_j])$ $(j = 0, 1)$ by the formula

$$\big(\widetilde{T}_j f\big)(t) = T\big(f(t)\big) \quad \text{for } f \in \mathcal{C}_j[X_j].$$

According to [9, Theorem 3.3] (with some trivial modifications), we have

$$\Gamma\big(\widetilde{T}_j\big) = \Gamma_j(T) \quad (j = 0, 1). \tag{3.1}$$

Now let $\varepsilon > 0$ be fixed and let $(y_\nu)_\nu$ be a sequence in $T(U_X)$. For each ν, $\exists\, x_\nu \in U_X$ with $y_\nu = T x_\nu$ and $\exists\, f_\nu \in \mathcal{A}_\theta(\mathbf{D}, \overline{X})$ such that $\|f_\nu\|_{\mathcal{A}_\theta} < 1$ and $x_\nu = \hat{f}_\nu(n)$. Then $g_\nu = T f_\nu \in \mathcal{H}^1_\theta(\mathbf{D}, \overline{Y})$ and $y_\nu = \hat{g}_\nu(n)$. Let $g_{j,\nu}$ denote the function $t \mapsto g_\nu(e^{it})$ for $t \in I^\theta_j$ $(j = 0, 1)$. Then $g_{j,\nu} \in \mathcal{L}^1_j[X_j]$. A similar argument in the proof of [9, Theorem 4.1] shows that one can find a sequence of integers $0 = k_1 < k_2 < \cdots$ and nonnegative numbers α^ν_l, for which $\sum_{l=k_\nu+1}^{k_{\nu+1}} \alpha^\nu_l = 1$, and the sequence of functions $(h_{j,\nu})_\nu$ $(j = 0, 1)$ given by

$$h_{j,\nu} = \sum_{l=k_\nu+1}^{k_{\nu+1}} \alpha^\nu_l g_{j,l}$$

satisfying

$$\|h_{j,k} - h_{j,m}\|_{\mathcal{L}^1_j} \leqslant \operatorname{csep}(h_{j,\nu})_\nu + \varepsilon. \tag{3.2}$$

Let $h_\nu = \sum_{l=k_\nu+1}^{k_{\nu+1}} \alpha^\nu_l g_l$. Then $\hat{h}_\nu(k) = \sum_{l=k_\nu+1}^{k_{\nu+1}} \alpha^\nu_l \hat{g}_l(k)$ for $0 \leqslant k \leqslant n$. Combining (1.6), (1.7), (3.1) and (3.2), we obtain

$$\begin{aligned}
\operatorname{csep}\Big(\big(y_\nu\big)_\nu\Big) &\leqslant \operatorname{csep}\Big(\big(\hat{h}_\nu(n)\big)_\nu\Big) \leqslant \big\|\hat{h}_1(n) - \hat{h}_2(n)\big\|_X \\
&\leqslant c\,\rho_{\theta,n}\Big(\|h_{0,1} - h_{0,2}\|_{\mathcal{L}^1_0}, \|h_{1,1} - h_{1,2}\|_{\mathcal{L}^1_1}\Big) \\
&\leqslant c\,\rho_{\theta,n}\bigg(\operatorname{csep}\Big(\big(h_{0,\nu}\big)_\nu\Big) + \varepsilon, \operatorname{csep}\Big(\big(h_{1,\nu}\big)_\nu\Big) + \varepsilon\bigg) \\
&\leqslant c\,\rho_{\theta,n}\Big(\Gamma\big(\widetilde{T}_0\big) + \varepsilon, \Gamma\big(\widetilde{T}_1\big) + \varepsilon\Big),
\end{aligned}$$

which implies that

$$\operatorname{csep}\Big(\big(y_\nu\big)_\nu\Big) \leqslant c\,\rho_{\theta,n}\big(\Gamma_0(T) + \varepsilon, \Gamma_1(T) + \varepsilon\big).$$

The estimate for $\Gamma_{\theta(n)}(T)$ follows by letting $\varepsilon \to 0$ and by taking the supremum on the left-hand side of the previous inequality. □

Next we turn to some inequalities concerning the spectral capacity, spectral radius and essential spectral radius of bounded linear operators under the $C_{\theta(\pm n)}$-methods. These inequalities follow from a spectral inclusion and some related results of the classical C_θ-methods, and the logarithmic convex estimates with parameter θ will be given. Let X be a Banach space, and let $T \in \mathcal{B}(X)$. We denote by $\operatorname{Sp}(T, X)$, $r(T, X)$ and $r_e(T, X)$ the spectrum, the spectral radius and the essential spectral radius of T, respectively. For a compact subset $\mathbf{K}$ of $\mathbf{C}$, the capacity of $\mathbf{K}$ is define by

$$\operatorname{Cap}\mathbf{K} = \inf_p \max_{z \in \mathbf{K}} |p(z)|^{1/\deg p}, \tag{3.3}$$

where the infimum is taken over all polynomials p with the leading coefficient equal to 1.

Proposition 3.2. *For $T \in \mathcal{B}(\overline{X})$ and $\theta \in (0,1)$, the inclusion*

$$\mathrm{Sp}\Big(T, C_{\theta(\pm n)}\big(\overline{X}\big)\Big) \subseteq \mathrm{Sp}\Big(T, C_\theta\big(\overline{X}\big)\Big) \tag{3.4}$$

holds. Consequently, we have

$$\mathrm{Cap}\,\mathrm{Sp}\left(T, C_{\theta(\pm n)}\big(\overline{X}\big)\right) \leqslant \mathrm{Cap}\,\mathrm{Sp}\left(T, X_0\right)^{1-\theta} \mathrm{Cap}\,\mathrm{Sp}\left(T, X_1\right)^{\theta}, \tag{3.5}$$

$$r\Big(T, C_{\theta(\pm n)}\big(\overline{X}\big)\Big) \leqslant r\big(T, X_0\big)^{1-\theta} r\big(T, X_1\big)^{\theta}, \tag{3.6}$$

and

$$r_e\Big(T, C_{\theta(\pm n)}\big(\overline{X}\big)\Big) \leqslant r_e\big(T, X_0\big)^{1-\theta} r_e\big(T, X_1\big)^{\theta}. \tag{3.7}$$

Proof. For inclusion (3.4), it is sufficient to show that, if $T \in \mathcal{B}(\overline{X})$ such that T is invertible on $C_\theta(\overline{X})$, then T is invertible on $C_{\theta(\pm n)}(\overline{X})$. We denote by T_θ^{-1} and $T_{\theta(\pm n)}^{-1}$ the inverses of T on $C_\theta(\overline{X})$ and $C_{\theta(\pm n)}(\overline{X})$, respectively. According to [2, Theorem 4], there exists $\varepsilon > 0$ such that, for $|\alpha - \theta| < \varepsilon$, T is invertible on $C_\alpha(\overline{X})$, and $T_\alpha^{-1} = T_\theta^{-1}$ on $\Delta\overline{X}$. Now we choose $0 < \theta_0, \theta_1, \zeta < 1$ such that $\theta_0 < \theta < \theta_1$, $\theta_1 - \theta_0 < \varepsilon$ and $\theta = (1-\zeta)\theta_0 + \zeta\theta_1$. Then

$$T_{\theta_j}^{-1} = T_\theta^{-1} \quad (j = 0, 1)$$

on $\Delta\overline{X}$, and

$$C_{\theta(\pm n)}\big(\overline{X}\big) = C_{\zeta(\pm n)}\Big(C_{\theta_0}\big(\overline{X}\big), C_{\theta_1}\big(\overline{X}\big)\Big)$$

by reiteration (1.2). It turns out that T is invertible on $C_{\theta(\pm n)}(\overline{X})$ for which $T_{\theta(\pm n)}^{-1} = T_\theta^{-1}$ on $\Delta\overline{X}$.

By (3.3) and (3.4), we have $r\Big(T, C_{\theta(\pm n)}\big(\overline{X}\big)\Big) \leqslant r\Big(T, C_\theta\big(\overline{X}\big)\Big)$ and

$$\mathrm{Cap}\,\mathrm{Sp}\left(T, C_{\theta(\pm n)}\big(\overline{X}\big)\right) \leqslant \mathrm{Cap}\,\mathrm{Sp}\left(T, C_{\theta(n)}\big(\overline{X}\big)\right).$$

This, together with [1, Lemma 3.1] and [2, Corollary 7], yields inequalities (3.5) and (3.6).

For inequality (3.7), we fix an arbitrary $\varepsilon > 0$, and put

$$r_j = r_e(T, X_j) + \varepsilon \quad \text{and} \quad \mathbf{K}_j = \left\{ z \in \mathbf{C} \,\middle|\, |z| \leqslant r_j \right\} \ (j = 0, 1).$$

Then

$$\Big(\mathrm{Sp}\big(T, X_0\big) - \mathbf{K}_0\Big) \cup \Big(\mathrm{Sp}\big(T, X_1\big) - \mathbf{K}_1\Big) = \{\lambda_1, \ldots, \lambda_l\}$$

is a finite set. Let now

$$c = \sup\left\{ \left| \Pi_{v=1}^{l}(z - \lambda_v) \right| \,\middle|\, |z| \leqslant r_0 \vee r_1 \right\},$$

and consider the polynomials $f_k(z) = z^k \Pi_{v=1}^{l}(z - \lambda_v)$ for $z \in \mathbf{C}$. By (3.6), we have

$$r\Big(f_k(t), C_{\theta(\pm n)}\big(\overline{X}\big)\Big) \leqslant r\big(f_k(T), X_0\big)^{1-\theta} r\big(f_k(T), X_1\big)^{\theta} \leqslant c\, r_0^{k(1-\theta)} r_1^{k\theta}.$$

Following the proof of [1, Theorem 3.3] word by word, we can complete the proof of (3.7). □

Finally, we show by some example that the estimates in Proposition 3.2 cannot improved for $n = 1$. Let F be an exact interpolation functor. As in [8, Definition 4.3], we denote q_F to be the spectral function of F in the following form:

$$q_F(t) = \sup\left\{ \|T\|_{F(\overline{X}),F(\overline{Y})} \,\middle|\, \|T\|_0 \vee \left(\|T\|_1 \big/ t\right) \leqslant 1 \right\},$$

where $\overline{X}$ and $\overline{Y}$ are arbitrary Banach couples. The semi-exponents of q_F are given by

$$\theta_0(q_F) = \lim_{t\to 0}\inf \frac{\log q_F(t)}{|\log t|} \quad \text{and} \quad \theta_\infty(q_F) = \lim_{t\to\infty}\sup \frac{\log q_F(t)}{|\log t|}.$$

Example [8, Example 9.5]. For $\alpha \geqslant 1$, let $\alpha\mathbf{T}$ be the circle in $\mathbf{C}$ with the center at origin and the radius equal to α, and let $\mathbf{W}$ be the annulus

$$\mathbf{W} = \left\{ z \in \mathbf{C} \,\middle|\, 1 \leqslant |z| \leqslant 2 \right\}.$$

Let now $\overline{X} = \overline{\mathcal{H}}^2(\mathbf{W})$ be the Banach couple consisting of the Hilbert spaces $\mathcal{L}^2(\mathbf{T}, dt)$ and $\mathcal{L}^2(2\mathbf{T}, dt)$ considered as subspaces of the dual of $\mathcal{H}^2(\mathbf{W})$. Thus, $\Delta\overline{X} = \mathcal{H}^2(\mathbf{W})$. Let $T \in \mathcal{B}(\overline{X})$ be the multiplication operator given by

$$Tf(z) = zf(z)$$

for any analytic function $f : \mathbf{W} \to \mathbf{C}$. Then

$$\operatorname{Sp}\left(T, \mathcal{H}^2(\mathbf{W})\right) = \mathbf{W}, \quad \operatorname{Sp}\left(T, \mathcal{L}^2(\alpha\mathbf{T})\right) = \alpha\mathbf{T}$$

and

$$r\left(T, \mathcal{L}^2(\alpha\mathbf{T})\right) = r_e\left(T, \mathcal{L}^2(\alpha\mathbf{T})\right) = \operatorname{Cap}\operatorname{Sp}\left(T, \mathcal{L}^2(\alpha\mathbf{T})\right) = \alpha$$

for $1 \leqslant \alpha \leqslant 2$. It is known that, if $F = C_\theta$, then $q_F(t) = t^\theta$. If $F = C_{\theta(\pm 1)}$, then $q_F = \rho_{\theta,1}$ by (1.1) and [7, Remark 5.7]. A straightforward calculation shows that

$$\theta_0(q_F) = \theta_\infty(q_F) = \theta$$

for $F = C_\theta$ or $C_{\theta(\pm 1)}$. It turns out that

$$\operatorname{Sp}\left(T, C_{\theta(\pm 1)}\left(\overline{X}\right)\right) = \operatorname{Sp}\left(T, C_\theta\left(\overline{X}\right)\right)$$

by [8, Theorem 9.1]. More precisely, $C_\theta\left(\mathcal{L}^2(\mathbf{T}), \mathcal{L}^2(2\mathbf{T})\right) = \mathcal{L}^2(2^\theta\mathbf{T})$. Therefore, the equalities in (3.5)–(3.7) hold true.

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